1. Error in Pb fractionation factor ($FPb_{25}$) for double spiked ($^{202}$Pb-$^{205}$Pb) samples

Because of the artificial nature of the $^{202}$Pb and $^{202}$Pb tracer isotopes, a unique algebraic expression for the linear lead fractionation factor as the coefficient per unit mass difference can be derived from the relationship:

$$R Pb_{25t} = R Pb_{25m} \cdot (1 - 3 \cdot F Pb_{25})$$

(1)

where $R Pb_{25m}$ is the measured $^{202}$Pb/$^{205}$Pb, and the tracer $^{202}$Pb/$^{205}$Pb, $R Pb_{25t}$, is by definition equal to the mass fractionation corrected ratio. Solving for $F Pb_{25}$ yields:

$$F Pb_{25} = \frac{1}{3} \left( 1 - \frac{R Pb_{25t}}{R Pb_{25m}} \right)$$

(2)

The error propagation equation for $F Pb_{25}$ may be written as (assuming all errors are uncorrelated):

$$\sigma_{F Pb_{25}}^2 = \left[ \left( \frac{\partial F Pb_{25}}{\partial R Pb_{25t}} \right) \cdot \sigma_{R Pb_{25t}} \right]^2 + \left[ \left( \frac{\partial F Pb_{25}}{\partial R Pb_{25m}} \right) \cdot \sigma_{R Pb_{25m}} \right]^2$$

(3)

The partial derivatives are calculated as:

$$\left( \frac{\partial F Pb_{25}}{\partial R Pb_{25t}} \right) = \left( \frac{1}{3 \cdot R Pb_{25m}} \right)$$

(4)

$$\left( \frac{\partial F Pb_{25}}{\partial R Pb_{25m}} \right) = \left( \frac{R Pb_{25t}}{3 \cdot (R Pb_{25m})^2} \right)$$

(5)

These partial derivatives and the appropriate variances can then be substituted into eqn. (3) to derive the uncertainty in the lead fractionation factor for double spiked ($^{202}$Pb-$^{205}$Pb) samples.
2. Error in U fractionation factor ($FU_{36}$) for double spiked ($^{233}$U-$^{236}$U) samples

Because of the artificial nature of the $^{233}$U and $^{236}$U tracer isotopes, a simple algebraic expression for the linear uranium fractionation factor, $FU_{36}$, as the coefficient per unit mass difference can be derived from the relationship:

$$RU_{36t} = RU_{36m} \cdot (1 - 3 \cdot FU_{36})$$  \hspace{1cm} (6)

where $RU_{36m}$ is the measured $^{233}$U/$^{236}$U, and the tracer $^{235}$U/$^{236}$U, $RU_{36t}$, is by definition equal to the mass fractionation corrected ratio. Solving for $FU_{36}$ yields:

$$FU_{36} = \frac{1}{3} \left(1 - \frac{RU_{36t}}{RU_{36m}}\right)$$  \hspace{1cm} (7)

The error propagation equation for $FU_{36}$ may be written as (assuming all errors are uncorrelated):

$$\sigma_{FU_{36}}^2 = \left[\left(\frac{\partial FU_{36}}{\partial RU_{36t}}\right) \cdot \sigma_{RU_{36t}}\right]^2 + \left[\left(\frac{\partial FU_{36}}{\partial RU_{36m}}\right) \cdot \sigma_{RU_{36m}}\right]^2$$  \hspace{1cm} (8)

The partial derivatives are calculated as:

$$\left(\frac{\partial FU_{36}}{\partial RU_{36t}}\right) = \left(\frac{1}{3 \cdot RU_{36m}}\right)$$  \hspace{1cm} (9)

$$\left(\frac{\partial FU_{36}}{\partial RU_{36m}}\right) = \left(\frac{RU_{36t}}{3 \cdot RU_{36m}^2}\right)$$  \hspace{1cm} (10)

These partial derivatives and the appropriate variances can then be substituted into eqn. (8) to derive the uncertainty in the lead fractionation factor for double spiked ($^{233}$U-$^{236}$U) samples.
3. $^{238}\text{U}$ sample from isotope dilution against $^{236}\text{U}$

First establishing the algebraic expression for sample $^{238}\text{U}$,

$$U^{238}s = [RU^{86}m \cdot (1 + 2 \cdot FU) \cdot (U^{236}t)] - (RU^{86}t \cdot U^{236}t) - U^{238}b$$ \hspace{1cm} (11)

the error propagation equation may be written as:

$$\sigma_{U^{238}s} = \left[ \left( \frac{\partial U^{238}s}{\partial RU^{86}m} \right) \cdot \sigma_{RU^{86}m} \right]^2 + \left[ \left( \frac{\partial U^{238}s}{\partial RU^{86}t} \right) \cdot \sigma_{RU^{86}t} \right]^2 + \left[ \left( \frac{\partial U^{238}s}{\partial U^{238}b} \right) \cdot \sigma_{U^{238}b} \right]^2 + \left[ \left( \frac{\partial U^{238}s}{\partial FU} \right) \cdot \sigma_{FU} \right]^2$$ \hspace{1cm} (12)

Note that to dramatically simplify our derivation, we are considering error correlations between the component terms of sample $^{238}\text{U}$ ($U^{238}s$) and the uranium fractionation factor ($FU$) to be trivial, which is justified given the contrasting dominant error sources in each quantity. The partial derivatives are calculated as follows:

$$\left( \frac{\partial U^{238}s}{\partial RU^{86}m} \right) = U^{236}t \cdot (1 + 2 \cdot FU)$$ \hspace{1cm} (13)

$$\left( \frac{\partial U^{238}s}{\partial RU^{86}t} \right) = -U^{236}t$$ \hspace{1cm} (14)

$$\left( \frac{\partial U^{238}s}{\partial U^{238}b} \right) = -1$$ \hspace{1cm} (15)

$$\left( \frac{\partial U^{238}s}{\partial FU} \right) = 2 \cdot U^{236}t \cdot RU^{86}m$$ \hspace{1cm} (16)
These partial derivatives and variances can then be substituted into eqn. 12 to derive the uncertainty in sample $^{238}$U.

4. $^{238}$U sample from isotope dilution against $^{233}$U

First establishing the algebraic expression for sample $^{238}$U,

$$U^{238}s = [RU^{83}m \cdot (1 + 5 \cdot FU) \cdot (U^{233}t)] - (RU^{83}t \cdot U^{233}t) - U^{238}b$$  \hspace{1cm} (17)$$

the error propagation equation may be written as:

$$\sigma_{U^{238}s} = \left[ \left( \frac{\partial U^{238}s}{\partial RU^{83}m} \right) \cdot \sigma_{RU^{83}m} \right]^2 + \left[ \left( \frac{\partial U^{238}s}{\partial RU^{83}t} \right) \cdot \sigma_{RU^{83}t} \right]^2$$

$$+ \left[ \left( \frac{\partial U^{238}s}{\partial U^{238}b} \right) \cdot \sigma_{U^{238}b} \right]^2 + \left[ \left( \frac{\partial U^{238}s}{\partial FU} \right) \cdot \sigma_{FU} \right]^2$$  \hspace{1cm} (18)$$

Note that to dramatically simplify our derivation, we are considering error correlations between the component terms of sample $^{238}$U ($U^{238}s$) and the uranium fractionation factor ($FU$) to be trivial, which is justified given the contrasting dominant error sources in each quantity. The partial derivatives are calculated as follows:

$$\left( \frac{\partial U^{238}s}{\partial RU^{83}m} \right) = U^{233}t \cdot (1 + 5 \cdot FU)$$  \hspace{1cm} (19)$$

$$\left( \frac{\partial U^{238}s}{\partial RU^{83}t} \right) = -U^{233}t$$  \hspace{1cm} (20)$$

$$\left( \frac{\partial U^{238}s}{\partial U^{238}b} \right) = -1$$  \hspace{1cm} (21)$$
\[
\left( \frac{\partial U^{238s}}{\partial FU} \right) = 5 \cdot U^{233t} \cdot RU^{83m} \tag{22}
\]

These partial derivatives and variances can then be substituted into eqn. 18 to derive the uncertainty in sample $^{238}\text{U}$. 